Exercise 1. *An integer is divisible by* 3 *if and only if the sum of its digits is divisible by* 3.

Proof. Let n be our integer, which we assume to be positive without any loss of generality, and suppose that n has the form

$$d_k d_{k-1} \cdots d_2 d_1 d_0$$

where the $d_i's$ are the *digits* of n, so each $d_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Observe now that we can express n in terms of the $d_i's$ mathematically as

$$\begin{aligned} n &= d_0 + d_1 \cdot 10 + d_2 \cdot 100 + \cdots + d_{k-1} \cdot 10^{k-1} + d_k \cdot 10^k \\ &= \sum_{i=0}^{k} d_i 10^i \end{aligned}$$

Let $s = \sum_{i=0}^{k} d_i$ denote the sum of the digits of n. Rewriting the statement we want to prove in this language we get

$$n = \sum_{i=0}^{k} d_i 10^i \text{ is divisible by } 3 \text{ if and only if } s = \sum_{i=0}^{k} d_i \text{ is divisible by } 3.$$

To prove this claim, we need to observe two facts:

(1) the number $10^i - 1$ is divisible by 3 for all $i = 0, 1, 2, ...$

(2) if $a = b + c$ and any two of a, b, c are divisible by a number d, then so must the third.

We will now do some manipulation to the sum for n:

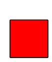
$$\begin{aligned} n &= \sum_{i=0}^{k} d_i 10^i = \sum_{i=0}^{k} d_i((10^i - 1) + 1) \\ &= \sum_{i=0}^{k} (d_i(10^i - 1) + d_i) = \sum_{i=0}^{k} d_i(10^i - 1) + \sum_{i=0}^{k} d_i \\ &= \sum_{i=0}^{k} d_i(10^i - 1) + s \end{aligned}$$

Let $a = \sum_{i=0}^{k} d_i(10^i - 1)$ and observe that we have written $n = a + s$. Notice also that a is always divisible by 3 since $10^i - 1$ is always divisible by 3 (using facts (1) and (2)). Now we can see that if n is divisible by 3, since a is divisible by 3, fact (2) tells us that s must also be divisible by 3. Similarly, if s is divisible by 3, again by fact (2), we get that a must also be divisible by 3. □

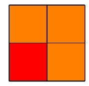
Exercise 2. *Derive and verify a closed form for the sum of n consecutive odd positive integers, i.e., for the sum*

$$1 + 3 + 5 + 7 + \cdots + (2n - 1).$$

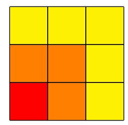
Proof. Let's start with 1, represented as a red square.



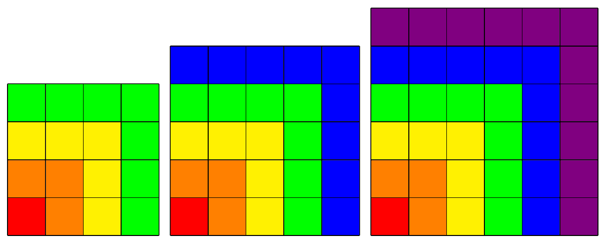
$1 + 3 = 4$, which we will illustrate by adding three orange squares to the picture with the red square.



Next, $1+3+5=9$, and we illustrate that by adding 5 yellow squares. Notice that we are able to keep the final shape as a square each time, and each of the sums is a square number!



The next three pictures are for $1+3+5+7$, $1+3+5+7+9$, and $1+3+5+7+9+11$, respectively.



These picture make pretty apparent that $1+3+5+7=16$, $1+3+5+7+9=25$, and $1+3+5+7+9+11=36$. Notice that the sum in each case is always the square of the number of things that we added, i.e., $1=1^2$, $1+3=2^2$, $1+3+5=3^2$, etc. So, then, it should be reasonable to make the *conjecture* that the sum of the first n odd numbers is

$$1+3+5+\cdots+(2n-1)=n^2.$$

Remark. *This is only a conjecture because, no matter how many examples we check, we can never check every single one by hand. The way around this is to use* induction. *With this process, we first verify a starting case is true, in this problem that case was $1=1^2$, then we assume that the property is still true at an arbitrary point in the sequence and try to prove that the next case must also be true. You can think of this process as having an infinitely long row of dominoes lined up. That starting case is verifying that we can push over the first domino. Then the* inductive step *is saying that if we were to look at any single domino anywhere in this row of dominoes, if that gets knocked over, then so must the domino after it. Putting these two things together, we know that each domino will always knock over the next one and since we started by knocking over the first one, all of the dominoes will eventually get knocked over.*

So, just how do we use induction to prove this? As we said earlier, we already know that the starting case, $1=1^2$, is true (this is putting $n=1$ into our conjecture formula). Now we will assume our *inductive hypothesis*, $1+3+5+\cdots+(2k-1)=k^2$, is true, and try to prove that $1+3+5+\cdots+(2k-1)+(2k+1)=(k+1)^2$ (this is replacing n with $k+1$ in our conjecture formula) is still true.

$$\begin{aligned}
1+3+5+\cdots+(2k-1)+(2k+1) &= (1+3+5+\cdots+(2k-1))+(2k+1) \\
&= k^2+(2k+1) && \text{(inductive hypothesis)} \\
&= k^2+2k+1 \\
&= (k+1)^2. && \text{(factor)}
\end{aligned}$$

Thus we have verified that

$$1+3+5+\cdots+(2k-1)+(2k+1)=(k+1)^2.$$

So, by induction, we have proved our conjecture formula to be true for all values of $n\geq 1$. $\square$

Lemma 1. *Let $x \in \mathbb{Z}$. If x^2 is even, then x is even.*

Proof. A direct proof here is difficult, so let's prove an equivalent statement, the contrapositive:

$$\text{Let } x \in \mathbb{Z}. \text{ If } x \text{ is odd, then } x^2 \text{ is odd.}$$

If x is odd, then there is an integer $n \in \mathbb{Z}$ such that $x = 2n+1$. Then $x^2 = (2n+1) = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$, so since $2n^2 + 2n \in \mathbb{Z}$, we have that x^2 is odd. Therefore, by contrapositive, we have the desired statement. $\square$

Theorem 1. *$\sqrt{2}$ is an irrational number.*

Proof. Suppose that $\sqrt{2}$ is rational. Then there are integers $a, b \in \mathbb{Z}$, $b \neq 0$, with $\sqrt{2} = \dfrac{a}{b}$ such that a and b have no common factors. Squaring both sides of the equation gives $2 = \dfrac{a^2}{b^2}$, or equivalently, $2b^2 = a^2$. Because $a^2 = 2b^2$, we have that a^2 is even. By the previous lemma, it follows that a must be even. Suppose that $a = 2c$, then plugging this into the equation gives $2b^2 = a^2 = (2c)^2 = 4c^2$. Dividing both sides by 2 gives $b^2 = 2c^2$. By the same logic as before, we find that b is an even number and therefore is divisible by 2. However, since a and b are both divisible by 2, this is a contradiction to our assumption. Therefore $\sqrt{2}$ is irrational. $\square$